Lecture 9: Linear (In)dependence continued (+ spanning sets)

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Recall:
Let $v_1, \ldots, v_m$ be elements of a vector space V.

i) $\{v_1, \ldots, v_m\}$ is linearly dependent (LD) if:
   $\exists\ \mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{m} \in \mathbb{R}$, not all 0, such that
   $a_1 v_1 + \cdots + a_m v_m = 0$

ii) $\{v_1, \ldots v_m\}$ is linearly independent (LI)if:
   $\{v_1, \ldots, v_m\}$ isn't LD <u>and</u>
   the only solution to $a_1 v_1 + \cdots + a_m v_m = 0$ is $\mathrm{a}_1 = \cdots = \mathrm{a}_\mathrm{m} = 0$

Hence:
u, v collinear if and only if {u,v} LD
u, v, w coplanar if and only if {u, v, w} LD

<u>7.5 Examples</u>

i) $\{(1,0), (0,1)\}$ LI
   because:
   $a(1,0) + b(0,1) = (0,0)$
   $a = 0$
   $b = 0$

ii) $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\} \subseteq \mathrm{M}_{22}(\mathbb{R})$ LD
   because:
   $1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + 1\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + 1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

iii) $\{x^2 + 4x + 4, x^2, x + 1\} \subseteq \mathbb{P}_2$ LD
   because:
   $1(x^2 + 4x + 4) - 1(x^2) - 4(x + 1) = 0$

<u>7.6 Observations</u>

1) $\{v\}$ LI if and only if $\mathrm{v} \neq 0$
   PROOF
   Firstly, for all $c \in \mathbb{R}$, $\mathrm{c} * 0 = 0$. Now, if v=0, then 1v=0, and {v} is LD. If v≠0, consider the equation kv=0. If there was a k≠0 solution, then:
   $v = \frac{1}{k} * k * v = \frac{1}{k} * 0 = 0$, which is a contradiction
   So, there can't be any such k, and {v} is LI.

2) $\{v_1, \ldots, v_m\}$ LD: any set containing $\{v_1, \ldots, v_m\}$ is LD

3) $\{v_1, \ldots, v_m\}$ LI: any subset of$\{v_1, \ldots, v_m\}$ is LI

4) {0} LD (special case of 1.)

5) Any set containing {0} is LD (special case of 2.)
   a. Note: this means that all subspaces are linearly dependent

6) {u,v} LD if and only if one vector is a multiple of the other.
   PROOF
   => au+bv=0 (a and b not both 0)
      if a≠0, then u= $-\frac{b}{a}v$
      if b≠0, then v= $-\frac{a}{b}u$
   <= if u=av, then $1 * u - a * v = 0, \{u, v\}$ LD

if v=au, then $a * u - 1 * v = 0, \{u, v\}$ LD

7) A set with more than 3 vectors can be LD even though no two are multiples of each other (look at example ii)

8) $\{v_1, \dots, v\_m\}$ LD if and only if $\exists$ k $\in \{1, \dots, \mathrm{m}\}$ such that $\mathrm{v_k} \in \mathrm{span}\{\mathrm{v_1}, \dots, \mathrm{v_{k-1}}, \mathrm{v_{k+1}}, \dots, \mathrm{v_m}\}$

PROOF

=> $a_1 v_1 + \cdots + a_m v_m = 0$ (not all $\mathrm{a_i} = 0$)

Choose k such that $a_k \neq 0$. Then,

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1} - \frac{a_{(k+1)}}{a_k} v_{k+1} - \cdots - \frac{a_m}{a_k} v_m$$

Hence, $v_k \in$ span $\{\mathrm{v_1}, \dots, \mathrm{v_{k-1}}, \mathrm{v_{k+1}}, \dots, \mathrm{v_m}\}$

<= $v_k = a_1 v_1 + \cdots + a_{(k-1)} v_{k-1} + a_{k+1} v_{k+1} + \cdots + a_m v_m$

=> $a_1 v_1 + \cdots + a_{k-1} v_{k-1} - 1 v_k + a_{k+1} v_{k+1} + \cdots + a_m v_m = 0$

(not all coefficients = 0)

8 Linear independency and spanning sets

8.1 Reducing LD spanning sets

Recall: span {(1,2), (2,4)} = span {(1,2)}

LD

to prove this, we observed that:

a(1,2)+b(2,4)

=a(1,2)+b*2(1,2)

=(a+2b)(1,2)

This can be generalized:

Theorem:

Consider $\mathrm{span}\{v_1, \dots, v_m\}$ If $v_1 \in \mathrm{span}\{\mathrm{v_2}, \dots, \mathrm{v_m}\}$, then:

$\mathrm{span}\{\mathrm{v_1}, \dots, \mathrm{v_m}\} = \mathrm{span}\{\mathrm{v_2}, \dots, \mathrm{v_m}\}$

PROOF

$\supseteq$ is clear. We give an argument for $\subseteq$:

$v_1 = a_2 v_2 + \cdots + a_m v_m$. Now, let w $\in \mathrm{span}\{\mathrm{v_1}, \dots, \mathrm{v_m}\}$.

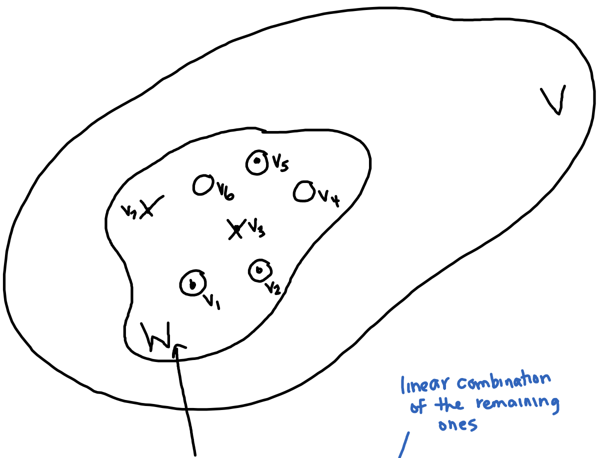
So, $w = b_1 v_1 + \cdots + b_m v_m$. We can rewrite that:

$w = b_1(a_2 v_2 + \cdots + a_m v_m) + b_2 v_2 + \cdots + b_m v_m$

$= (b_2 + b_1 a_2) v_2 + \cdots + (b_m + b_1 a_m) v_m$

$\in \mathrm{span}\{\mathrm{v_2}, \dots, \mathrm{v_m}\}$

We can reduce any LD spanning set.



linear combination of the remaining ones

$w = \mathrm{span}\{\mathrm{v_1}, \mathrm{v_2}, \mathrm{v_3}, \mathrm{v_4}, \mathrm{v_5}, \mathrm{v_6}, \mathrm{v_7}\}$

$= \mathrm{span}\{\mathrm{v_1}, \mathrm{v_2}, \quad \mathrm{v_4}, \mathrm{v_5}, \mathrm{v_6}, \mathrm{v_7}\}$

$= \mathrm{span}\underbrace{\{\mathrm{v_1}, \mathrm{v_2}, \mathrm{v_4}, \mathrm{v_5}, \mathrm{v_6}\}} +$

linear combination of the remaining ones

$= \text{span}\{v_1, v_2, \quad v_4, v_5, v_6, v_7\}$

linear combination of the remaining ones

$= \text{span}\underbrace{\{v_1, v_2, v_4, v_5, v_6\}}_{\text{eventually LI}} +$